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A Discrete Artificial Boundary Condition for Steady Incompressible Viscous Flows in a No-Slip Channel Using a Fast Iterative Method

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We design a discrete artificial boundary condition for the steady incompressible Navier–Stokes equations in stream function vorticity formulation in an infinite channel. The new boundary condition is derived from a linearized Navier–Stokes system and a fast iterative method. Numerical experiments for the nonlinear Navier–Stokes equations are presented. The discrete artificial boundary condition is compared to Dirichlet and Neumann boundary conditions for the flow over a forward or backward facing step and for flow past a rectangular cylinder. Numerical results show that our boundary condition is very effective. © 1994 Academic Press, Inc.

1. INTRODUCTION

Many numerical simulations of viscous fluid flow problems are given on unbounded domains, such as fluid flow around obstacles and fluid flow in a channel. One difficulty in these problems is the unboundedness of the physical domain. In engineering, the usual method is to introduce an artificial boundary to reduce these problems to a bounded computational domain and to set up an artificial boundary condition at the artificial boundary, such as Neumann or Dirichlet boundary conditions for the dependent variable. In general, the above artificial boundary conditions are only very rough approximations of the exact boundary condition at the artificial boundary. When high accuracy is required, the bounded computational domain must be quite large, and the cost of the computation is increased. In practice, in order to limit the computational cost, the artificial boundary must be chosen not too far from the domain of interest. During the last 10 years ways to design artificial boundary conditions with high accuracy on a given artificial boundary or solving partial differential equations on an unbounded domain have been studied often. For instance, Goldstein [1] studied Helmholtz-type equations in waveguides and other unbounded domains. The problem was replaced by a boundary value problem on a bounded computational domain. The behavior of the solution near infinity is incorporated in a nonlocal boundary condition. Feng [2] designed the asymptotic radiation conditions for the reduced wave equation. Han and Wu [3, 4] presented the exact boundary conditions at an artificial boundary for the Laplace equation and the linear elasticity system; moreover, a sequence of approximations to the exact boundary condition at the artificial boundary was given. The exact boundary condition at an artificial boundary for partial differential equations in a cylinder was obtained by Hagstrom and Keller [5]. Shortly thereafter, they used this technique to solve nonlinear problems [6]. A family of artificial boundary conditions for unsteady Oseen equations in the velocity pressure formulation with small viscosity was developed by Halpern [7], Halpern and Schatzman [8], which was then applied to unsteady Navier–Stokes (N–S) equations. Nataf [9] designed an open boundary condition for steady Oseen equations in stream-function vorticity formulation, which is applied to viscous incompressible fluid flow around a body in a flat channel with slip boundary conditions on the wall.

Recently Hagstrom [10, 11] proposed asymptotic boundary conditions at artificial boundaries for the simulation of time-dependent fluid flows and applied them to solve nonlinear N–S equations.

In this paper we consider a steady viscous incompressible fluid flow around a body in a no-slip flat channel defined by $\{-\infty < x < \infty \text{ and } 0 \leqslant y \leqslant L\}$. In a region sufficiently far from the body, the flow is almost Poiseuille flow, equal to $u_\infty(y) = \alpha y(L-y)$ and $v_\infty = 0$, α is a positive constant, in which N–S equations can be linearized to a system of linear N–S equations, but they are not Oseen equations. In this case the techniques developed in [7–9] cannot be applied directly, because the coefficients of the linear N–S equations are not constants. The purpose of this paper is to design a discrete artificial boundary condition for the linear N–S equations in stream-function vorticity formulation and to apply it to numerical simulations of the steady incompressible viscous fluid flow in a channel.

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2. A SYSTEM OF LINEAR NAVIER–STOKES EQUATIONS

In this paper we consider the numerical simulation of a steady incompressible viscous flow arounding a body (domain Ω_i) in a no-slip channel defined by $\Re \times [0, L]$. Let u, v, and p denote velocity and presure, then u, v, and p satisfy the following N–S equations in domain $\Omega = \Re \times (0, L) \backslash \bar{\Omega}_i$:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = \nu \Delta u, \tag{2.1}$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = \nu \Delta v, \tag{2.2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.3}$$

and boundary conditions

$$u|_{y=0,L} = v|_{y=0,L} = 0, \qquad -\infty < x < +\infty, \tag{2.4}$$

$$u|_{\partial\Omega_i} = v|_{\partial\Omega_i} = 0 \tag{2.5}$$

$$\begin{aligned} &u(x, y) \to u_\infty(y) = \alpha y(L-y), \\ &v(x, y) \to v_\infty = 0, \qquad \text{when} \quad x \to \pm\infty, \end{aligned} \tag{2.6}$$

where $\nu > 0$ is the kinematic viscosity.

Taking two constants $b < c$, such that $\bar{\Omega}_i \subset (b, c) \times (0, L)$, then Ω is divided by the artificial boundaries $\Gamma_b = \{x = b, 0 \leqslant y \leqslant L\}$ and $\Gamma_c = \{x = c, 0 \leqslant y \leqslant L\}$ into three parts, Ω^b, Ω^T, and Ω^c, with

$$\begin{aligned} \Omega^b &= \{(x, y) \mid -\infty < x < b, 0 < y < L\}, \\ \Omega^T &= \{(x, y) \mid b < x < c, 0 < y < L\} \backslash \bar{\Omega}_i, \\ \Omega^c &= \{(x, y) \mid c < x < +\infty, 0 < y < L\}. \end{aligned}$$

When $|b|$, c are sufficiently large, in the domain $\Omega^b \cup \Omega^c$ the flow is almost Poiseuille flow. So N–S equations (2.1)–(2.3) can be linearized, namely on domain Ω^c (and Ω^b), the solutions u, v, and p of the problem (2.1)–(2.6) approximately satisfy the linear N–S equations,

$$u_\infty(y) \frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = \nu \Delta u, \qquad \text{in } \Omega^c, \tag{2.7}$$

$$u_\infty(y) \frac{\partial v}{\partial x} + \frac{\partial p}{\partial y} = \nu \Delta v, \qquad \text{in } \Omega^c, \tag{2.8}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad \text{in } \Omega^c, \tag{2.9}$$

and the boundary conditions

$$u|_{y=0,L} = v|_{y=0,L} = 0, \qquad c < x < +\infty, \tag{2.10}$$

$$\begin{aligned} &u(x, y) \to u_\infty(y), \\ &v(x, y) \to 0, \qquad \text{when} \quad x \to +\infty. \end{aligned} \tag{2.11}$$

Let ψ and ω denote the stream-function and vorticity, then

$$\frac{\partial \psi}{\partial y} = u, \qquad \frac{\partial \psi}{\partial x} = -v, \tag{2.12}$$

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \tag{2.13}$$

and Eqs. (2.1)–(2.3), and the boundary condition (2.4)–(2.6) are reduced to

$$u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} - \nu \Delta \omega = 0, \qquad \text{in } \Omega, \tag{2.14}$$

$$\Delta \psi + \omega = 0, \qquad \text{in } \Omega, \tag{2.15}$$

$$\psi|_{y=0} = 0, \qquad \psi|_{y=L} = \psi_L \equiv \int_0^L u_\infty(s)\, ds, \qquad -\infty < x < +\infty \tag{2.16}$$

$$\left.\frac{\partial \psi}{\partial y}\right|_{y=0,L} = 0, \qquad -\infty < x < +\infty \tag{2.17}$$

$$\psi = \text{const}, \qquad \frac{\partial \psi}{\partial n} = 0, \qquad \text{on } \partial\Omega_i \tag{2.18}$$

$$\begin{aligned} &\psi \to \psi_\infty(y) \equiv \int_0^y u_\infty(s)\, ds, \\ &\omega \to \omega_\infty(y) \equiv -u'_\infty(y), \end{aligned} \tag{2.19}$$

where $\partial/\partial n$ denotes the outward normal derivative.

Furthermore, linear N–S equations (2.7)–(2.9) and boundary conditions (2.10)–(2.11) are reduced to

$$u'_\infty(y) \frac{\partial^2 \psi}{\partial x\, \partial y} - u_\infty(y) \frac{\partial \omega}{\partial x} + \nu \Delta \omega = 0, \qquad \text{in } \Omega^c, \tag{2.20}$$

$$\Delta \psi + \omega = 0, \qquad \text{in } \Omega^c, \tag{2.21}$$

$$\psi|_{y=0} = 0, \qquad \psi|_{y=L} = \psi_L, \qquad c < x < +\infty \tag{2.22}$$

$$\left.\frac{\partial \psi}{\partial y}\right|_{y=0,L} = 0, \qquad c < x < +\infty \tag{2.23}$$

$$\begin{aligned} &\psi \to \psi_\infty(y), \\ &\omega \to \omega_\infty(y), \qquad \text{when} \quad x \to +\infty \end{aligned} \tag{2.24}$$

Since the boundary condition on the artificial boundary Γ_c is unknown, the problem (2.20)–(2.24) is an incompletely posed problem. It cannot be solved. Let

$$\psi|_{x=c}=\psi_c(y), \qquad \omega|_{x=c}=\omega_c(y), \qquad 0\leqslant y\leqslant L. \tag{2.25}$$

For given functions $\psi_c(y)$ with $\psi_c(0)=0$, $\psi_c(L)=\psi_L$, $(d\psi_c/dy)|_{y=0,L}=0$, and $\omega_c(y)$, we discuss the numerical solution of problem (2.20)–(2.25) and design a discrete artificial boundary condition on the line Γ_c for the problem (2.14)–(2.19).

3. A DISCRETE ARTIFICIAL BOUNDARY CONDITION

We now consider the finite-difference approximations of problem (2.20)–(2.25). Let Δx and $\Delta y = L/N$ be the two mesh sizes, where N is a positive integer. The domain Ω^c is discretized by $\{(x_j = c + j\,\Delta x,\ y_k = k\,\Delta y),\ j=0, 1, 2, ..., k= 0, 1, 2, ..., N\}$. The following finite difference scheme is used to solve problem (2.20)–(2.25),

$$\begin{aligned}
&\frac{u'_\infty(y_k)}{4\,\Delta x\,\Delta y}\left[\psi_{j+1,k+1}-\psi_{j+1,k-1}-\psi_{j-1,k+1}+\psi_{j-1,k-1}\right]\\
&\quad-\frac{u_\infty(y_k)}{2\,\Delta x}\left[\omega_{j+1,k}-\omega_{j-1,k}\right]\\
&\quad+v\left[\frac{\omega_{j+1,k}-2\omega_{j,k}+\omega_{j-1,k}}{\Delta x^2}\right.\\
&\quad\left.+\frac{\omega_{j,k+1}-2\omega_{j,k}+\omega_{j,k-1}}{\Delta y^2}\right]=0,
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
&\frac{\psi_{j+1,k}-2\psi_{j,k}+\psi_{j-1,k}}{\Delta x^2}+\frac{\psi_{j,k+1}-2\psi_{j,k}+\psi_{j,k-1}}{\Delta y^2}\\
&\quad+\omega_{j,k}=0, \qquad 1\leqslant k\leqslant N-1, \qquad j=1, 2, ...,
\end{aligned} \tag{3.2}$$

with boundary conditions

$$\psi_{j,0}=0, \qquad \psi_{j,N}=\psi_L, \qquad j=0, 1, 2, ..., \tag{3.3}$$

$$\begin{aligned}
&\omega_{j,0}=-\frac{1}{2}\omega_{j,1}+\frac{3(\psi_{j,0}-\psi_{j,1})}{\Delta y^2},\\
&\omega_{j,N}=-\frac{1}{2}\omega_{j,N-1}+\frac{3(\psi_{j,N}-\psi_{j,N-1})}{\Delta y^2},\\
&\qquad j=0, 1, 2, ...,
\end{aligned} \tag{3.4}$$

$$\lim_{j\to+\infty}\psi_{j,k}=\psi_\infty(y_k), \qquad \lim_{j\to+\infty}\omega_{j,k}=\omega_\infty(y_k), \tag{3.5}$$

$$\begin{aligned}
&\psi_{0,k}=\psi_c(y_k), \qquad \omega_{0,k}=\omega_c(y_k),\\
&\qquad k=1, 2, ..., N-1.
\end{aligned} \tag{3.6}$$

Let

$$X_j=[\omega_{j,1}, ..., \omega_{j,N-1}, \psi_{j,1}, ..., \psi_{j,N-1}]^{\mathrm{T}},$$
$$X_\infty=[\omega_\infty(y_1), ..., \omega_\infty(y_{N-1}), \psi_\infty(y_1), ..., \psi_\infty(y_{N-1})]^{\mathrm{T}},$$

then Eqs. (3.1)–(3.6) are equivalent to the system of linear algebraic equations including infinitely many unknowns:

For given $X_0\in\mathfrak{R}^{2N-2}$, find $\{X_1, X_2, ...\}$, such that

$$\begin{aligned}
&A_0X_{j-1}+B_0X_j+C_0X_{j+1}=D_0, \qquad j=1, 2, ...,\\
&\lim_{j\to+\infty}X_j=X_\infty,
\end{aligned} \tag{3.7}$$

where

$$A_0=\begin{pmatrix}
\alpha_1 & 0 & 0 & \cdots & 0 & 0 & -\delta_1 & 0 & \cdots & 0\\
0 & \alpha_2 & 0 & \cdots & 0 & \delta_2 & 0 & -\delta_2 & \cdots & 0\\
0 & 0 & \alpha_3 & \cdots & 0 & 0 & \delta_3 & 0 & \cdots & 0\\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots\\
0 & 0 & \cdots & 0 & \alpha_{N-1} & 0 & 0 & \cdots & \delta_{N-1} & 0\\
0 & 0 & 0 & \cdots & 0 & \eta_x & 0 & 0 & \cdots & 0\\
0 & 0 & 0 & \cdots & 0 & 0 & \eta_x & 0 & \cdots & 0\\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \eta_x & \cdots & 0\\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots\\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \eta_x
\end{pmatrix}$$

$$B_0=\begin{pmatrix}
\beta_1 & v\eta_y & 0 & \cdots & 0 & -3v\eta_y^2 & 0 & 0 & \cdots & 0\\
v\eta_y & \beta_2 & v\eta_y & \cdots & 0 & 0 & 0 & 0 & \cdots & 0\\
0 & v\eta_y & \beta_3 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0\\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots\\
0 & 0 & \cdots & v\eta_y & \beta_{N-1} & 0 & 0 & \cdots & 0 & -3v\eta_y^2\\
1 & 0 & 0 & \cdots & 0 & \eta_{xy} & \eta_y & 0 & \cdots & 0\\
0 & 1 & 0 & \cdots & 0 & \eta_y & \eta_{xy} & \eta_y & \cdots & 0\\
0 & 0 & 1 & \cdots & 0 & 0 & \eta_y & \eta_{xy} & \cdots & 0\\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots\\
0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & \eta_y & \eta_{xy}
\end{pmatrix}$$

$$C_0=\begin{pmatrix}
\gamma_1 & 0 & 0 & \cdots & 0 & 0 & \delta_1 & 0 & \cdots & 0\\
0 & \gamma_2 & 0 & \cdots & 0 & -\delta_2 & 0 & \delta_2 & \cdots & 0\\
0 & 0 & \gamma_3 & \cdots & 0 & 0 & -\delta_3 & 0 & \cdots & 0\\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots\\
0 & 0 & \cdots & 0 & \gamma_{N-1} & 0 & 0 & \cdots & -\delta_{N-1} & 0\\
0 & 0 & 0 & \cdots & 0 & \eta_x & 0 & 0 & \cdots & 0\\
0 & 0 & 0 & \cdots & 0 & 0 & \eta_x & 0 & \cdots & 0\\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \eta_x & \cdots & 0\\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots\\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \eta_x
\end{pmatrix}$$

with

$$\alpha_k = \frac{\nu}{\Delta x^2} + \frac{u_\infty(y_k)}{2\,\Delta x}, \qquad k = 1, 2, ..., N-1,$$

$$\delta_k = \frac{u'_\infty(y_k)}{4\,\Delta x\,\Delta y}, \qquad k = 1, 2, ..., N-1,$$

$$\beta_1 = -\frac{2\nu}{\Delta x^2} - \frac{5\nu}{2\,\Delta y^2},$$

$$\beta_k = -\frac{2\nu}{\Delta x^2} - \frac{2\nu}{\Delta y^2}, \qquad k = 2, ..., N-2,$$

$$\beta_{N-1} = -\frac{2\nu}{\Delta x^2} - \frac{5\nu}{2\,\Delta y^2},$$

$$\eta_x = \frac{1}{\Delta x^2}, \qquad \eta_y = \frac{1}{\Delta y^2}, \qquad \eta_{xy} = -2(\eta_x + \eta_y),$$

$$\gamma_k = \frac{\nu}{\Delta x^2} - \frac{u_\infty(y_k)}{2\,\Delta x}, \qquad k = 1, 2, ..., N-1,$$

and

$$D_0 = [d_1, d_2, ..., d_{2N-2}]^{\mathrm{T}}.$$

with

$$d_k = 0, \qquad k = 1, 2, ..., N-2, N, ..., 2N-3,$$

$$d_{N-1} = -\frac{3\nu}{\Delta y^4}\psi_L, \qquad d_{2N-2} = -\frac{1}{\Delta y^2}\psi_L.$$

At first we present a special solution of the problem (3.7).
Let

$$(A_0 + B_0 + C_0)\,X_\infty = G \equiv [g_1, g_2, ..., g_{2N-2}]^{\mathrm{T}}.$$

A computation shows that

$$g_1 = -\frac{5\nu}{2\,\Delta y^2}\omega_\infty(y_1) + \frac{\nu}{\Delta y^2}\omega_\infty(y_2) - \frac{3\nu}{\Delta y^4}\psi_\infty(y_1),$$

$$g_k = \frac{\nu}{\Delta y^2}\omega_\infty(y_{k-1}) - \frac{2\nu}{\Delta y^2}\omega_\infty(y_k) + \frac{\nu}{\Delta y^2}\omega_\infty(y_{k+1}),$$
$$2 \leqslant k \leqslant N-2,$$

$$g_{N-1} = -\frac{5\nu}{2\,\Delta y^2}\omega_\infty(y_{N-1}) + \frac{\nu}{\Delta y^2}\omega_\infty(y_{N-2}) - \frac{3\nu}{\Delta y^4}\psi_\infty(y_{N-1}),$$

$$g_N = \omega_\infty(y_1) + \frac{-2\psi_\infty(y_1) + \psi_\infty(y_2)}{\Delta y^2},$$

$$g_{N-1+k} = \omega_\infty(y_k) + \frac{\psi_\infty(y_{k-1}) - 2\psi_\infty(y_k) + \psi_\infty(y_{k+1})}{\Delta y^2},$$
$$2 \leqslant k \leqslant N-2,$$

$$g_{2N-2} = \omega_\infty(y_{N-1}) + \frac{-2\psi_\infty(y_{N-1}) + \psi_\infty(y_{N-2})}{\Delta y^2}.$$

Since $\psi_\infty(y)$ is a polynomial of degree three, $\omega_\infty(y)$ is a polynomial of degree one and $\psi''_\infty(y) + \omega_\infty(y) = 0$, $\psi_\infty(0) = \psi'_\infty(0) = \psi'_\infty(L) = 0$, $\psi_\infty(L) = \psi_L$, so we obtain

$$g_k = 0, \qquad k = 1, 2, ..., N-2,$$

$$g_{N-1} = -\frac{3\nu}{\Delta y^4}\psi_\infty(y_N) = -\frac{3\nu}{\Delta y^4}\psi_L,$$

$$g_{N-1+k} = 0, \qquad k = 1, 2, ..., N-2,$$

$$g_{2N-2} = -\frac{\psi_\infty(y_N)}{\Delta y^2} = -\frac{\psi_L}{\Delta y^2}.$$

Hence we have

$$(A_0 + B_0 + C_0)\,X_\infty = D_0. \tag{3.8}$$

From Eq. (3.8), we know that $\{X_j = X_\infty, j = 1, 2, ...\}$ is a solution of problem (3.7) with $X_0 = X_\infty$.
Let

$$Y_j = X_j - X_\infty, \qquad j = 0, 1, ..., \tag{3.9}$$

then the problem (3.7) can be rewritten as follows: For given $Y_0 \in \Re^{2N-2}$, find $\{Y_j, j = 1, 2, ...\}$ such that

$$\begin{aligned} &A_0 Y_{j-1} + B_0 Y_j + C_0 Y_{j+1} = 0, \qquad j = 1, 2, ..., \\ &\lim_{j \to +\infty} Y_j = 0. \end{aligned} \tag{3.10}$$

From [12] we know that there are three methods for solving the problem (3.10). One of them is called the direct method, which reduces the problem (3.10) to a eigenvalue problem of a $(4N-4) \times (4N-4)$ matrix. In order to obtain the solution of problem (3.10), we must compute all eigenvalues and corresponding eigenvectors of the matrix. We prefer to use the fast iterative method [12] to solve the problem (3.10), which is cheaper. In (3.10) for $j-1$ and $j+1$ we have

$$Y_{j-1} = -B_0^{-1}A_0 Y_{j-2} - B_0^{-1}C_0 Y_j, \qquad j = 2, 3, ..., \tag{3.11}$$

$$Y_{j+1} = -B_0^{-1}A_0 Y_j - B_0^{-1}C_0 Y_{j+2}, \qquad j = 0, 1, ... \tag{3.12}$$

Substituting (3.11) and (3.12) into (3.10), we obtain

$$A_1 Y_{j-2} + B_1 Y_j + C_1 Y_{j+2} = 0, \qquad j = 2, 3, ...,$$

with

$$\begin{aligned} A_1 &= -A_0 B_0^{-1} A_0, \\ B_1 &= B_0 - A_0 B_0^{-1} C_0 - C_0 B_0^{-1} A_0, \\ C_1 &= -C_0 B_0^{-1} C_0. \end{aligned}$$

This procedure can be repeated. After k ($k = 1, 2, ...$) iterations, we have

$$A_k Y_{j-2^k} + B_k Y_j + C_k Y_{j+2^k} = 0, \qquad j = 2^k, 2^k + 1, ..., \tag{3.13}$$

with

$$\begin{aligned} A_k &= -A_{k-1} B_{k-1}^{-1} A_{k-1}, \\ B_k &= B_{k-1} - A_{k-1} B_{k-1}^{-1} C_{k-1} - C_{k-1} B_{k-1}^{-1} A_{k-1}, \\ C_k &= -C_{k-1} B_{k-1}^{-1} C_{k-1}, \end{aligned} \tag{3.14}$$

and, furthermore, from (3.13) we have

$$\begin{aligned} Y_j = -B_k^{-1} A_k Y_{j-2^k} - B_k^{-1} C_k Y_{j+2^k}, \qquad & k = 0, 1, 2, ..., \\ j = 2^k, 2^k + 1, ... \qquad & \end{aligned} \tag{3.15}$$

We now return to the problem (3.10). Consider the matrix equation

$$A_0 Y_0 + B_0 Y_1 + C_0 Y_2 = 0. \tag{3.16}$$

Inserting (3.15) with $k = 1, j = 2^k$ into (3.16) and eliminating Y_2, we obtain

$$\tilde{A}_1 Y_0 + B_0 Y_1 + \tilde{C}_1 Y_{2^1+1} = 0 \tag{3.17}$$

with

$$\begin{aligned} \tilde{A}_1 &= A_0 - C_0 B_1^{-1} A_1, \\ \tilde{C}_1 &= -C_0 B_1^{-1} C_1. \end{aligned} \tag{3.18}$$

Inserting (3.15) with $k = 2, j = 2^k$ into (3.17) and eliminating Y_{2^2}, we have

$$\tilde{A}_2 Y_0 + B_0 Y_1 + \tilde{C}_2 Y_{2^2+1} = 0,$$

with

$$\begin{aligned} \tilde{A}_2 &= \tilde{A}_1 - \tilde{C}_1 B_2^{-1} A_2, \\ \tilde{C}_2 &= -\tilde{C}_1 B_2^{-1} C_2. \end{aligned}$$

Continue this procedure. After k time steps, we obtain

$$\tilde{A}_k Y_0 + B_0 Y_1 + \tilde{C}_k Y_{2^k+1} = 0 \tag{3.19}$$

with

$$\begin{aligned} \tilde{A}_k &= \tilde{A}_{k-1} - \tilde{C}_{k-1} B_k^{-1} A_k, \\ \tilde{C}_k &= -\tilde{C}_{k-1} B_k^{-1} C_k. \end{aligned} \tag{3.20}$$

Supposing that the limits exist, let $\tilde{A}_\infty = \lim_{k \to \infty} \tilde{A}_k$, $\tilde{C}_\infty = \lim_{k \to \infty} \tilde{C}_k$. In (3.19) letting k go to infinity we obtain

$$\tilde{A}_\infty Y_0 + B_0 Y_1 = 0.$$

Furthermore, we have

$$X_1 = -T_\infty X_0 + (T_\infty + I) X_\infty \tag{3.21}$$

with $T_\infty = B_0^{-1} \tilde{A}_\infty$, where I is the unit matrix.

For given integer k ($k = 1, 2, ...$), we can obtain a sequence of approximate relationships between Y_0 and Y_1:

$$\tilde{A}_k Y_0 + B_0 Y_1 = 0.$$

Moreover, we have

$$X_1 = -T_k X_0 + (I + T_k) X_\infty \tag{3.22}$$

with $T_k = B_0^{-1} \tilde{A}_k$, $k = 1, 2, ...$

Let $W = [\partial\omega(c, y_1)/\partial x, ..., \partial\omega(c, y_{N-1})/\partial x, \partial\psi(c, y_1)/\partial x, ..., \partial\psi(c, y_{N-1})/\partial x]^{\mathrm{T}}$, then approximately we obtain

$$X_1 = X_0 + \Delta x W. \tag{3.23}$$

Substituting (3.23) into (3.21) and (3.22) we obtain the following discrete artificial boundary conditions on artificial boundary Γ_c:

$$W = -\frac{1}{\Delta x} (T_\infty + I)(X_0 - X_\infty), \tag{3.24}$$

$$W = -\frac{1}{\Delta x} (T_k + I)(X_0 - X_\infty), \qquad k = 1, 2, ... \tag{3.25}$$

In a similar manner, we can obtain discrete artificial boundary conditions on the boundary $\Gamma_b = \{x = b, 0 \leqslant y \leqslant L\}$.

4. NUMERICAL IMPLEMENTATION AND EXAMPLES

We now consider the numercial solution of the original problem (2.14)–(2.19) on the given computational domain Ω^T. This steady state solution is computed as the limit in time of the unsteady N–S equations, which are made discrete by an ADI method [13]. The inflow boundary conditions

$$\psi(b, y) = \psi_\infty(y), \qquad \omega(b, y) = \omega_\infty(y), \qquad 0 \leqslant y \leqslant L, \tag{4.1}$$

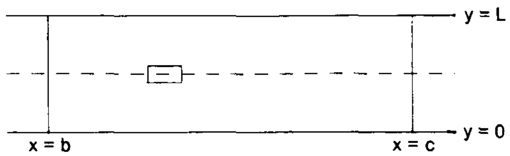

FIGURE 1

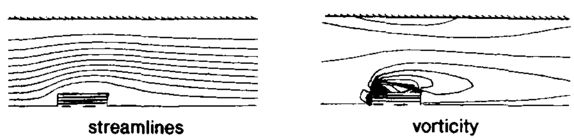

FIGURE 2

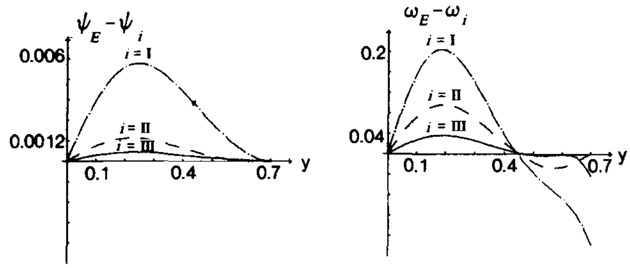

FIGURE 3

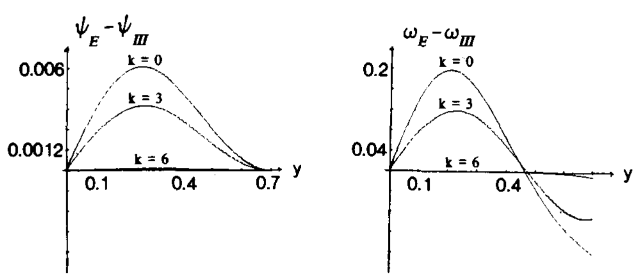

FIGURE 4

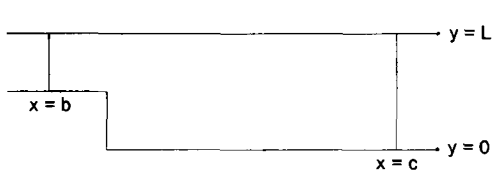

FIGURE 5

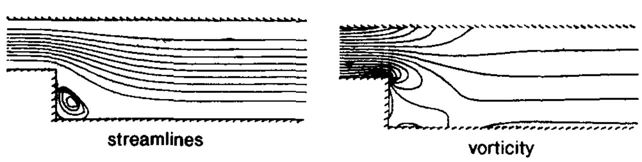

FIGURE 6

TABLE I

Errors	$i=\mathrm{I}$	$i=\mathrm{II}$	$i=\mathrm{III}$
ω	0.480768	0.204270	0.087274
ψ	0.013783	0.003086	0.001290

are prescribed on the artificial boundary Γ_b. On the artificial boundary Γ_c, the following three different types of outflow boundary conditions on ω and ψ are used in each example for comparison:

Type I. Dirichlet boundary condition,

$$\psi(c, y) = \psi_\infty(y), \qquad \omega(c, y) = \omega_\infty(y);$$

Type II. Neumann boundary condition,

$$\frac{\partial \psi}{\partial x}(c, y) = 0, \qquad \frac{\partial \omega}{\partial x}(c, y) = 0;$$

Type III. Discrete artificial boundary condition (3.25) or (3.22) with varied k.

In each example, the results are compared with an "exact solution." This solution is obtained by using an outflow boundary very far from the obstacle or step and by using Neumann boundary conditions on this outflow boundary. To be precise, the distance between the inflow boundary and the outflow boundary for the "exact solution" is twice the distance in our example.

EXAMPLE 1. Flow in a horizontal channel with a rectangular cylinder obstacle, as shown in Fig. 1. The obstacle is defined by the domain

$$\Omega_i = \{(x, y) \mid 1.2 < x < 1.6, \tfrac{3}{7}L < y < \tfrac{4}{7}L\}.$$

The bounded computational domain Ω^T is given by

$$\Omega^T = \{(x, y) \mid b < x < c, 0 < y < L\} \backslash \bar{\Omega}_i$$

and $u_\infty(y) = (4u_\infty/L^2)\, y(L-y)$; hence $\psi_\infty(y)$ and $\omega_\infty(y)$ are given by

$$\omega_\infty(y) = \frac{4u_\infty}{L^2}(2y - L),$$

$$\psi_\infty(y) = \frac{4u_\infty}{L^2}\, y^2\left(\frac{L}{2} - \frac{y}{3}\right).$$

TABLE II

Norms	$k=0$	$k=2$	$k=4$	$k=6$	$k=8$
$\|\tilde{A}_k - \tilde{A}_\infty\|$	0.210E + 02	0.466E + 00	0.110E − 02	0.125E − 12	0.135E − 48
$\|\tilde{C}_k\|$	0.122E + 03	0.153E + 01	0.612E − 02	0.321E − 11	0.245E − 48

We take $b=0$, $c=4.4$, $L=1.4$, $u_\infty=1.0$, $\nu=0.01$, and $\mathrm{Re}=1/\nu=100$.

Let (ω_E, ψ_E) denote the "exact solution" (Fig. 2) and (ω_i, ψ_i) $(i=\mathrm{I, II, III})$ denote the numerical solutions corresponding the boundary condition types I, II, and III on the artificial boundary Γ_c. The error $\omega_E-\omega_i$, $\psi_E-\psi_i$ on the boundary Γ_c are given in Fig. 3. Let

$$\mathrm{err}(f_E-\tilde{f}_i)=\left\{\sum_{j=1}^{N}(f_E(c, y_j)-\tilde{f}_i(c, y_j))^2\right\}^{1/2}$$

then the errors $\mathrm{err}(\omega_E-\omega_i)$, $\mathrm{err}(\psi_E-\psi_i)$ are given in Table I.

In the practical computation, the integers k in formula (3.22) or (3.25) are taken by $k=6$. The influence of k is shown in Fig. 4.

EXAMPLE 2. Backward-facing step flow (Fig. 5). The bounded computational domain is given by

$$\Omega^T=\left\{(x, y)\,|\,b<x\leqslant b+\frac{L}{2}, \frac{L}{2}<y<L;\right.$$
$$\left. b+\frac{L}{2}\leqslant x<c, 0<y<L\right\}.$$

The boundary conditions on $\Gamma_b=\{x=b, b/2\leqslant y\leqslant L\}$ is given by

$$\omega(b, y)=\frac{16u_\infty}{L^2}(4y-3L),$$
$$\psi(b, y)=\frac{8u_\infty}{3L^2}\left(y-\frac{L}{2}\right)^2(5L-4y).$$

The boundary conditions on $\Gamma_c=\{x=c, 0\leqslant y\leqslant L\}$ are the same as in Example 1.

The constants b, c, L, u_∞, and Re are given by $b=0$, $c=2.1$, $L=0.7$, $u_\infty=0.25$, and $\mathrm{Re}=100$. The comparisons of the "exact solution" ω_E, ψ_E (Fig. 6) and the numerical solutions (ω_i, ψ_i) $(i=\mathrm{I, II, III})$ are given in Fig. 7. The influence of the location of the artificial boundary Γ_c is shown in Fig. 8.

Table II shows that when $k\to+\infty$, the matrix sequence $\{\tilde{A}_k\}$, $\{\tilde{C}_k\}$ converges to $\tilde{A}_\infty$ and $\tilde{C}_\infty=0$ rapidly. The norm $\|A\|$ is defined by

$$\|A\|=\max_{1\leqslant i\leqslant 2N-2,\, 1\leqslant j\leqslant 2N-2}|a_{ij}|.$$

EXAMPLE 3. Forward-facing step flow (Fig. 9). The bounded computational domain Ω^T is given by

$$\Omega^T=\{(x, y)\,|\,b<x\leqslant b+3L, -L<y<L;$$
$$b+3L\leqslant x<c, 0<y<L\}.$$

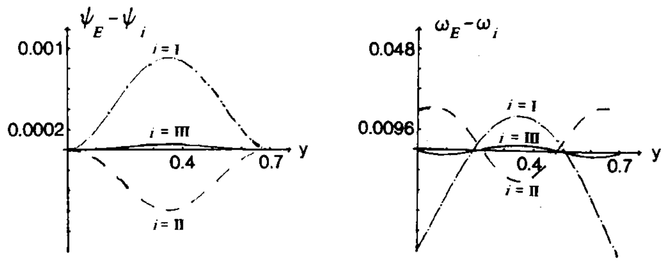

FIGURE 7

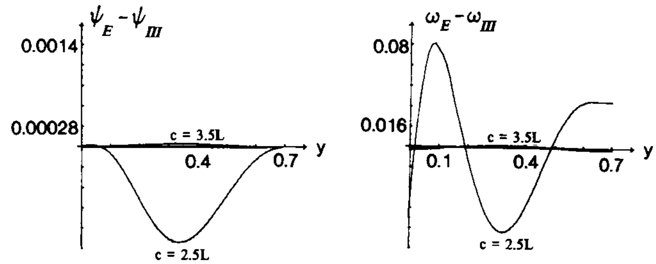

FIGURE 8

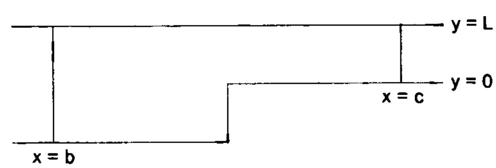

FIGURE 9

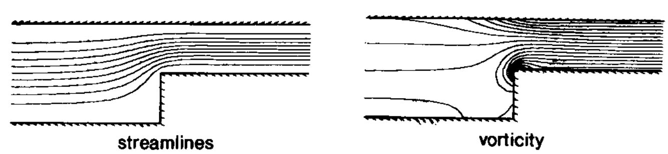

FIGURE 10

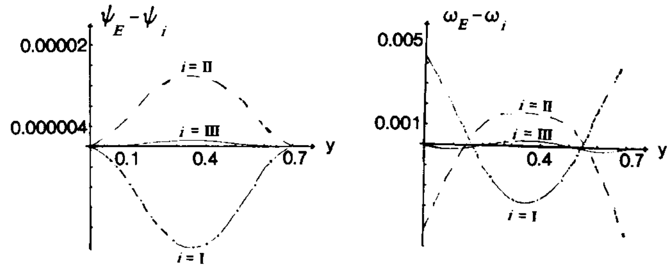

FIGURE 11

TABLE III

Errors	$i=\mathrm{I}$	$i=\mathrm{II}$	$i=\mathrm{III}$
ω	1.2311	0.4648	0.4190
ψ	0.0047	0.0027	0.0018

On the boundary $\Gamma_b = \{x=b,\ -L \leqslant y \leqslant L\}$, the boundary conditions are given by

$$\omega(b, y) = \frac{u_\infty}{L^2}\, y,$$

$$\psi(b, y) = \frac{u_\infty}{6L^2}(y+L)^2(2L-y).$$

The boundary conditions on Γ_c are the same as in Example 1. We take $b=0$, $c=2.1$, $L=0.35$, $u_\infty=0.5$, $v=0.01$, and $\mathrm{Re}=100$. The comparison of the "exact solution" (Fig. 10) ω_E, ψ_E, and the numerical solutions (ω_i, ψ_i) $(i=\mathrm{I, II, III})$ on the artificial boundary Γ_c are given in Fig. 11.

EXAMPLE 4. The effect of using the new condition at the inflow. We consider the backward-facing step flow as shown in the Fig. 5. The bounded computational domain and the Neumann boundary conditions on Γ_c are the same as those in Example 2. We take $b=0$, $c=2.1$, $L=0.7$, $u_\infty=1$, and $\mathrm{Re}=100$. In this example, the three different types boundary conditions at inflow boundary Γ_b are used in our computation. We obtain three solutions (ω_i, ψ_i) $(i=\mathrm{I, II, III})$. The errors $\mathrm{err}(\omega_E-\omega_i)$ and $\mathrm{err}(\psi_E-\psi_i)$ on the boundary Γ_b are shown in Table III where (ω_E, ψ_E) is the "exact solution." From Table III we can see the new boundary condition is the best one. But the effect of using different type boundary conditions at the inflow boundary is not sensitive.

The examples show that the discrete artificial boundary condition presented in this paper is very effective. Especially, the convergence of iterative method is fast. Furthermore, this approach can be applied to many problems of incompressible viscous flows.

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